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Accurate Values of the Exponent Governing Potential Flow about Semi-Infinite Cones

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IT is well known that, for axisymmetric, inviscid, and incompressible flow about a semi-infinite cone, the velocity V on the surface of the cone varies with the distance s from the vertex in the following manner:

$$V = Cs^m \tag{1}$$

where C is simply a scaling constant. The flow is thus completely characterized by the exponent m, which is a function of Θ, the semivertex angle of the cone. Tabulated values of m do not appear to be readily available in the literature. Reference 1 gives a small graph of m vs Θ and also lists three references, including the original work of Ref. 2. However, none of the three is contained in ordinary technical libraries.

The equation relating m and Θ is

$$P_{m+1}'(-\cos\Theta) = 0 \tag{2}$$

where the prime denotes differentiation and the function P_{m+1} is the Legendre function that remains finite when its argument equals unity for all values of its order $m + 1$ (see Ref. 3). Using the series expansion of P_{m+1}, this equation was solved numerically for m at values of Θ ranging from 0° to 180° by 5° increments. The results are given in Table 1.

Table 1 Values of the exponent governing potential flow about semi-infinite cones

Θ, deg	m	Θ, deg	$1/m$
0	0.0000000	90	1.0000000
5	0.0037441	95	0.8779641
10	0.0145329	100	0.7715075
15	0.0316314	105	0.6779398
20	0.0544316	110	0.5951432
25	0.0825162	115	0.5214293
30	0.1156458	120	0.4554368
35	0.1537334	125	0.3960580
40	0.1968232	130	0.3423826
45	0.2450773	135	0.2936569
50	0.2987690	140	0.2492515
55	0.3582834	145	0.2086375
60	0.4241237	150	0.1713675
65	0.4969244	155	0.1370604
70	0.5774709	160	0.1053903
75	0.6667277	165	0.0760764
80	0.7658769	170	0.0488761
85	0.8763705	175	0.0235785
90	1.0000000	180	0.0000000

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Variable Collision Frequency Effects on Hall-Current Accelerator Characteristics

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Nomenclature

B = magnetic induction
b = ion-slip parameter
E = electric field
e = charge on electron
J = current density
L = reference length
M = Mach number
m = mass flow
m_a = atom mass
N = interaction parameter
P = pressure
S = parameter defined by Eq. (4)
T = temperature
V = velocity
z' = dimensionless axial coordinate = z/L
α = degree of ionization
β = Hall parameter defined by Eq. (5)
γ = specific heat ratio
η = energy conversion efficiency
ρ = mass density
σ = electrical conductivity
τ = time between collisions
$\bar{\tau}_e$ = mean electron collision time
ϕ = voltage difference between inlet and exit
ω = cyclotron frequency

Subscripts

ei = electron-ion collision
ea = electron-atom collision
ia = ion-atom collision
0 = accelerator inlet
z = axial direction
θ = azimuthal direction

ONE-dimensional magnetogasdynamic (MGD) analyses of the coaxial Hall-current accelerator have recently been presented by Brandmaier, Durand, Gourdine, and Rubel[1] and by Cann and Marlotte.[2] Each considered the steady continuum flow of an ideal, electrically neutral, slightly ionized, three-species gas mixture through a narrow constant

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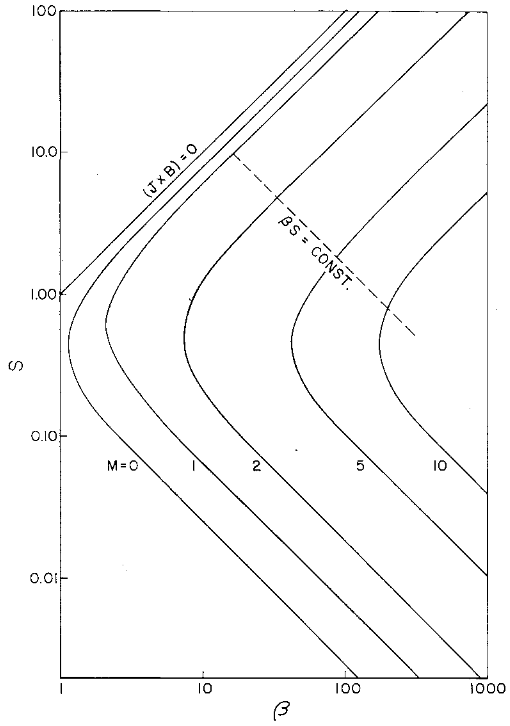

Fig. 1 Supersonic acceleration region.

area annular channel under the influence of a constant radial magnetic field and an axial electric field. In integrating the applicable differential equations, constant values of $\omega_e\bar{\tau}_e$, the ratio of electron cyclotron frequency to the mean collision frequency of electrons with neutrals and ions, and b, the ion-slip parameter that contains the effect of the ratio of ion-cyclotron frequency to ion-neutral collision frequency, were assumed. However, the analysis in Ref. 1 assumed a constant electrical conductivity, whereas the analysis in Ref. 2 assumed a constant degree of ionization. The purpose of this note is to extend the analysis of Ref. 1 to include the effects of the variation of $\omega_e\bar{\tau}_e$ and b with collision frequency during acceleration at a constant degree of ionization.

The differential equations for the axial velocity and Mach number are[1]

$$\frac{1}{V_z}\frac{d}{dz'}V_z = -\gamma N \frac{1}{S^2}\frac{M^2}{M^2-1}\left[(S^2+1) - \frac{1}{\gamma}(\beta S+1)\right] \quad (1)$$

$$\frac{1}{M}\frac{d}{dz'}M = -\gamma N \frac{1}{S^2}\frac{M^2}{M^2-1}\left[(S^2+1)\times\left(1+\frac{\gamma-1}{2}M^2\right) - \frac{\gamma+1}{2\gamma}(\beta S+1)\right] \quad (2)$$

In Eqs. (1) and (2)

$$N = \frac{\sigma B^2 L}{m(1+b)(1+\beta^2)} = \frac{\alpha e}{ma}\frac{BL}{V_z}\frac{\beta}{1+\beta^2} \quad (3)$$

$$S = V_z B/(E - V_\theta B) \quad (4)$$

and

$$\beta = \omega_e\bar{\tau}_e/(1+b) \quad (5)$$

The ion-slip parameter is given by

$$b = 2(1-\alpha)^2\omega_i\tau_{ia}\omega_e\bar{\tau}_e \quad (6)$$

For supersonic acceleration and an increasing Mach number, the sign of the bracket in both Eqs. (1) and (2) must be negative. Thus, for sonic conditions at the accelerator inlet this requirement is satisfied in the region to the right of the $M = 1$ curve in the S-β plane shown in Fig. 1. Similar curves for $M > 1$ can be determined from Eq. (2) by equating the bracket to zero and solving for S as a function of β and M.

The axial current density can be expressed as[1]

$$J_z = \frac{\alpha e}{m_a} m \frac{\beta^2}{\beta^2+1}\left(1+\frac{1}{\beta S}\right) \quad (7)$$

Considering constant α and $\beta^2 \gg 1$, βS must be constant to satisfy charge conservation. As a result, acceleration occurs along a path defined by βS = const. This path is shown in Fig. 1 for supersonic acceleration from Mach 1.

A relation between β and V_z was established from Eqs. (5) and (6) by considering the mean collision time variation. A mean electron temperature that was consistent with a constant degree of ionization was assumed. Thus τ_{ea} (see Ref. 3) is inversely proportional to density. This also applies to τ_{ei} since the Coulomb logarithm is relatively insensitive to density variation.[4] Similarly, at low ratios of electric field to pressure, ion mobility and therefore τ_{ia} are also inversely proportional to density.[5] As a result β increases with decreasing density until b increases to 1 after which it decreases. Utilizing the βS = const and ρV_z = const conditions, the V_z-S relation is

$$\frac{V_z}{V_{z_0}} = \left(\frac{1+b_0}{2b_0}\right)\frac{S}{S_0} - \left[\left(\frac{1+b_0}{2b_0}\frac{S}{S_0}\right)^2 - \frac{1}{b_0}\right]^{1/2} \quad (8)$$

where the subscript 0 refers to inlet conditions. Acceleration continues until the bracket in Eq. (8) is zero, corresponding to $b = 1$, or

$$(S/S_0) = [2/(1+b_0)]b_0^{1/2} \quad (9)$$

The maximum velocity ratio is, therefore,

$$(V_z/V_{z_0}) = b_0^{-1/2} \quad (10)$$

Equation (10) clearly demonstrates that, for supersonic acceleration from Mach 1 to the high exit velocities required for space propulsion, the initial ion-slip parameter must be much less than 1.

Equations (1, 2, and 8) require a numerical solution for arbitrary b_0. However a closed form solution was found for $b_0 = 0$, in which case $V_z/V_{z_0} = S_0/S$. For $M_0 = 1$ the re-

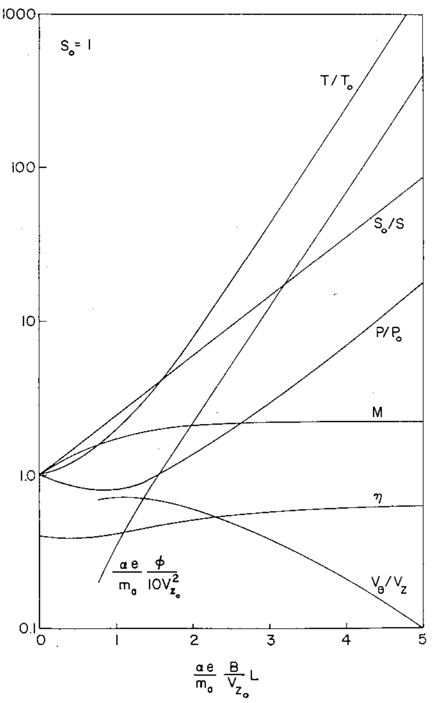

Fig. 2 Accelerator parameters for $S_0 = 1$.

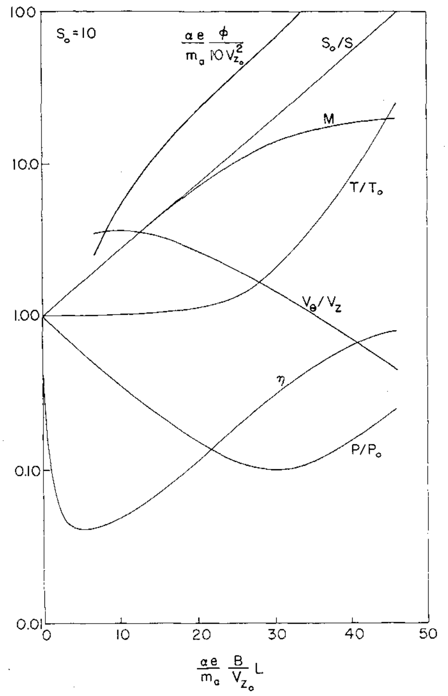

Fig. 3 Accelerator parameters for $S_0 = 10$.

sulting equations for Mach number, voltage, accelerator length, and azimuthal velocity in terms of (S_0/S) are

$$M^2 = \frac{1 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)}{(S/S_0)^2 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)} \tag{11}$$

$$\frac{\alpha e}{m_a}\phi = \frac{1}{2}V_{z_0}^2\left(1 + \frac{\gamma - 1}{\gamma}\frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1} \times \left\{\left[\left(\frac{S_0}{S}\right)^2 - 1\right] + (S_0^2 + 1) \times \left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\left(\ln\frac{S_0}{S}\right)^2\right\} \tag{12}$$

$$\frac{\alpha e}{m_a}\frac{B}{V_{z_0}}z = S_0\left(1 + \frac{\gamma - 1}{\gamma}\frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\ln\frac{S_0}{S} \tag{13}$$

and

$$\frac{V_\theta}{V_z} = S_0\left(1 + \frac{1}{S_0^2}\right)\left(1 + \frac{\gamma - 1}{\gamma + 1}\frac{1}{S_0^2}\right)^{-1}\frac{S}{S_0}\ln\frac{S_0}{S} \tag{14}$$

At $(S/S_0) = 0$, corresponding to an infinite velocity ratio, the Mach number is finite and depends only on S_0. For $b_0 > 0$, however, both velocity ratio and Mach number are finite. Constant Mach number contours are shown in Fig. 1 for $b_0 = 0$. The linear dependence of V_θ on z can be obtained by eliminating (S_0/S) from Eqs. (13) and (14).

Finally an integrated conversion efficiency, representing the ratio of axial body force work to electrical energy input[1] was found to be

$$\eta = \frac{\{1 + [(\gamma - 1)/\gamma](1/S_0^2)\}[(S_0/S)^2 - 1] - (1/\gamma)\ln(S_0/S)^2}{[1 + (1/S_0^2)]\{[(S_0/S)^2 - 1] + (1 + S_0^2) \times \{1 + [(\gamma - 1)/(\gamma + 1)](1/S_0^2)\}^{-1}[\ln(S_0/S)]^2\}} \tag{15}$$

At S_0, $\eta = (\gamma - 1/\gamma)$. It can be shown from the electron energy equation that the remaining $(1/\gamma)$ of the electrical energy input appears as initial "electron heating."[6] Inspection of Eq. (15) indicates that given (S_0/S), there is an optimum value of S_0 that maximizes η. As (S_0/S) increases, the optimum S_0 increases. This is associated with the ratio of azimuthal to axial accelerating force which can be expressed as[1]

$$\frac{J_z B}{J_\theta B} = \frac{1 + \beta S}{\beta - S} \tag{16}$$

Thus $J_z B > J_\theta B$ for $S > (\beta - 1)/(\beta + 1)$ and the major portion of the body force work appears as rotational energy. This is apparent from Figs. 2 and 3, which show the variation of accelerator parameters with length for $\gamma = \frac{5}{3}$ and $S_0 = 1$ and 10, respectively. For $S_0 = 10$, η is low until S decreases to appreciably less than 1, whereas for $S_0 = 1$, there is almost a steady rise until a limiting efficiency determined by Joule heating is reached.

The constant degree of ionization analysis described in this note corresponds to negligibly small ionization and recombination rates. Actual accelerator characteristics can be bracketed by considering infinite rates in which case, the degree of ionization is in equilibrium with the local pressure, ion-atom temperature, and electron temperature. This requires consideration of the electron energy equation[6] and a suitable Saha-type equation.[7] Since α would increase due to the static temperature rise shown in Figs. 2 and 3, βS would increase during acceleration according to Eq. (7).

As $\alpha \to 1$ the form of Ohm's Law used to derive Eqs. (1, 2, and 7) must be modified to include electron pressure gradients. These gradients act to decouple the thermodynamic from the body force effects, particularly at large ratios of electron to ion-atom temperatures.

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Matrizant of Keplerian Motion

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In a recent paper[1] on the matrizant of Keplerian motion (or the "error matrix" or "guidance matrix," etc.), I was rash enough to comment that there was no general formula applicable to every type of orbit, except over time spans short enough for series expansions in the time to be used. I would have done better to have written that I was not, at the time, aware of any such formula, for I rapidly became indebted to W. H. Goodyear[2] for sending me one. Goodyear's formulas

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